Math 201 — Fall 2011–12
Calculus and Analytic Geometry III, all sections
Quiz 1, October 22 — Duration: 90 minutes

GRADES:

1 (/18)	2 (/15)	3 (/17)	4 (/14)	5 (/14)	6 (/12)	7 (/10)	TOTAL	GRADE

YOUR NAME: Key

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1 MWF 3, Kobeissi Recitation F 11	Section 2 MWF 3, Kobeissi Recitation F 5	Section 3 MWF 3, Kobeissi Recitation F 4	Section 4 MWF 3, Kobeissi Recitation F 10
Section 5 MWF 10, Abi-Khuzam Recitation T 11	Section 6 MWF 10, Abi-Khuzam Recitation T 3:30	Section 7 MWF 10, Abi-Khuzam Recitation T 5	Section 8 MWF 10, Abi-Khuzam Recitation T 2
Section 9 MWF 11, Brock Recitation T 12:30	Section 10 MWF 11, Brock Recitation T 2	Section 11 MWF 11, Brock Recitation T 11	Section 12 MWF 11, Brock Recitation T 3:30
Section 13 MWF 2, Nahlus Recitation Th 11	Section 14 MWF 2, Nahlus Recitation Th 3:30	Section 15 MWF 2, Nahlus Recitation Th 8	Section 16 MWF 2, Nahlus Recitation Th 5
Section 17 MWF 8, Makdisi Recitation F 2	Section 18 MWF 8, Makdisi Recitation Th 8	Section 19 MWF 8, Makdisi Recitation Th 2	Section 20 MWF 8, Makdisi Recitation Th 3:30
Section 21 MWF 1, Raji Recitation M 8	Section 22 MWF 1, Raji Recitation M 9	Section 23 MWF 1, Raji Recitation M 4	
Section 24 MWF 10, Egeileh Recitation F 11	Section 25 MWF 10, Egeileh Recitation F 2	Section 26 MWF 10, Egeileh Recitation F 3	

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside this white booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit.
3. You may also use the back of any page for solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Closed book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

1. (6 pts each part, 18 pts total) For each of the following sequences, find, with justification, the limit, or else explain briefly why the limit does not exist.

(a)

$$\lim_{n\to\infty}\left(\frac{n^{10}-2n^9}{n^{10}+n^9}\right)^n$$

$$= \lim_{n\to\infty} \frac{\left(1-\frac{2}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n} = \frac{e^{-2}}{e^1} = e^{-3}.$$

(b)

$$\lim_{n\to\infty} 3^{1/n}\cdot\sin\left(\frac{1}{n}\right)\cdot\cos(n^2+1)$$

$0 \le \left| 3^{\frac{1}{n}} \cdot \sin\frac{1}{n} \cdot \cos(n^2+1) \right| \le 3^{1/n} \cdot \sin\frac{1}{n}$ since $\sin\frac{1}{n} > 0$ for $n = 1, 2, 3, \ldots$

But $\lim\limits_{n\to\infty} 3^{1/n}\sin\frac{1}{n} = 1 \cdot 0 = 0$

Hence $\lim\limits_{n\to\infty} 3^{1/n} \cdot \sin\frac{1}{n}\cos(n^2+1) = 0$ by the Sandwich theorem

(c)

$$\lim_{n\to\infty}\cos n\pi$$

$\cos n\pi = (-1)^n$.

So $\cos 2n\pi = 1$ while $\cos(2n+1)\pi = -1$.

$\therefore$ the sequence does not converge.

2. (5 pts each part, 15 pts total) For each of the series in (a) and (b), determine whether it converges or diverges. For the series in part (c), find all values of x for which the series is convergent. Justify your answers.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2 \cos n}{3^{n+1}}$$

$\sum_{n=1}^{\infty} \left| \frac{n^2 \cos n}{3^{n+1}} \right| \le \sum_{n=1}^{\infty} \frac{n^2}{3^{n+1}}$.

Applying the root test ~~for~~ the series $\sum_1^{\infty} \frac{n^2}{3^{n+1}}$ we get

$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{3^{n+1}}} = \lim_{n\to\infty} \frac{(n^{1/n})^2}{3 \cdot 3^{1/n}} = \frac{(1)^2}{3 \cdot 1} = \frac{1}{3} < 1.$

$\therefore$ The series $\sum_1^{\infty} \frac{n^2}{3^{n+1}}$ converges. $\therefore$ the series $\sum_1^{\infty} \frac{n^2 \cos n}{3^{n+1}}$ converges absolutely. Hence $\sum_1^{\infty} \frac{n^2 \cos n}{3^{n+1}}$ converges.

(b)

$$\sum_{n=1}^{\infty} (n^{1/n} - 1)$$

Since $n^{1/n} - 1 = e^{\frac{1}{n}\ln n} - 1 = \frac{1}{n}\ln n + \frac{1}{2!}\left(\frac{\ln n}{n}\right)^2 + \dots$

we compare this series with $\sum_1^{\infty} \frac{\ln n}{n}$.

Noticing that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, we apply L.C.T to get

$\lim_{n\to\infty} \left(\frac{n^{1/n} - 1}{\frac{\ln n}{n}}\right) = \lim_{n\to\infty} \frac{e^{\frac{\ln n}{n}} - 1}{\frac{\ln n}{n}} = \lim_{x\to 0} \frac{e^x - 1}{x} = 1$. So the two series behave alike.

But the series $\sum_1^{\infty} \frac{\ln n}{n}$ diverges because $\sum_2^{\infty} \frac{\ln n}{n} \ge (\ln 2) \sum_2^{\infty} \frac{1}{n}$.

$\therefore$ the series $\sum_1^{\infty} (n^{1/n} - 1)$ diverge.

(c) In this part find all values of x for which the series is convergent.

$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!} (x-3)^n$$

$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(2n+2)!} (x-3)^{n+1}}{\frac{n!}{(2n)!} (x-3)^n} \right| = \lim_{n\to\infty} |x-3| \cdot \frac{(n+1) \cdot (2n)!}{(2n+2)!}$

$= \lim |x-3| \cdot \frac{(n+1) \cdot (2n)!}{(2n)! (2n+1)(2n+2)} = \lim \frac{|x-3|}{2(2n+1)} = 0$

for all x

$\therefore$ The series converges absolutely, and so converge for

$-\infty < x < \infty$

3. (10 pts part (a), 7 pts part (b), 17 pts total)

(a) Compute the n^{th} partial sum S_n of the following series, and use it to find, according to the value of c, the sum of the series.

$$\sum_{k=1}^{\infty} \frac{c^{k+1} - c^k}{(c^k+1)(c^{k+1}+1)}, \qquad c > 0.$$

$$S_n = \sum_{k=1}^{n} \frac{c^{k+1} - c^k}{(c^k+1)(c^{k+1}+1)} = \sum_{k=1}^{n} \left(\frac{1}{c^k+1} - \frac{1}{c^{k+1}+1} \right)$$

$$= \frac{1}{c+1} - \frac{1}{c^{n+1}+1}.$$

If $0 \le c < 1$, then $\lim\limits_{n \to \infty} c^{n+1} = 0$, and $\lim S_n = \frac{1}{c+1} - \frac{1}{0+1} = \frac{-c}{c+1}$.

If $c = 1$, then $\lim\limits_{n \to \infty} c^{n+1} = 1$, and $\lim S_n = 0$.

If $c > 1$, then $\lim c^{n+1} = \infty$ and $\lim S_n = \frac{1}{c+1} - 0 = \frac{1}{c+1}$.

(b) Find the limit of the following sequence:

$$\lim_{n \to \infty} \frac{\frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots + \frac{1}{n \ln n}}{\ln(\sqrt{\ln n})}$$

The function $\frac{\ln x}{x}$ is positive and decreasing for $x \ge 2$. So

$$\int_3^{n+1} \frac{dx}{x \ln x} < \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \cdots + \frac{1}{n \ln n} < \int_2^{n} \frac{dx}{x \ln x}$$

Letting $u = \ln x$, $du = \frac{1}{x}dx$, $\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u = \ln \ln x$.

$\therefore \ln(\ln(n+1)) - \ln\ln 3 < \frac{1}{3\ln 3} + \cdots + \frac{1}{n \ln n} < \ln\ln n - \ln\ln 2$

since $\ln(n+1) > \ln n$,

$\therefore \ln(\ln n) - \ln\ln 3 < \frac{1}{3\ln 3} + \cdots + \frac{1}{n \ln n} < \ln\ln n - \ln\ln 2$,

$$\therefore \frac{\ln(\ln n) - \ln\ln 3}{\ln\sqrt{\ln n}} < \frac{\frac{1}{3\ln 3} + \cdots + \frac{1}{n\ln n}}{\ln\sqrt{\ln n}} < \frac{\ln\ln n - \ln\ln 2}{\ln\sqrt{\ln n}}$$

$$2\left(\frac{\ln\ln n - \ln\ln 3}{\ln\ln n}\right) < \frac{\frac{1}{3\ln 3} + \cdots + \frac{1}{n\ln n}}{\ln\sqrt{\ln n}} < 2\left(\frac{\ln\ln n - \ln\ln 2}{\ln\ln n}\right)$$

Hence $\lim\limits_{n \to \infty} \frac{\frac{1}{3\ln 3} + \cdots + \frac{1}{n\ln n}}{\ln\sqrt{\ln n}} = 2$.

4. **(14 pts)** Let $f(x) = \dfrac{1}{x^2 - 7x - 8}$. Find the Taylor series expansion of f about $a = 3$, (i.e., centered at $a = 3$), and use it to find $f^{(n)}(3)$.

$$f(x) = \frac{1}{x^2-7x-8} = \frac{1}{(x-8)(x+1)} = \frac{1/9}{x-8} - \frac{1/9}{x+1}$$

$$= \frac{1/9}{(x-3)-5} - \frac{1/9}{(x-3)+4} = \frac{\frac{-1}{45}}{1-\left(\frac{x-3}{5}\right)} - \frac{1/36}{1+\left(\frac{x-3}{4}\right)}$$

$$= -\frac{1}{45}\sum_{n=0}^{\infty}\left(\frac{x-3}{5}\right)^n - \frac{1}{36}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-3}{4}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[-\frac{1}{45}\cdot\frac{1}{5^n} - \frac{(-1)^n}{36}\cdot\frac{1}{4^n}\right](x-3)^n$$

which is valid for all $|x-3|<4$.

$$\therefore \frac{f^{(n)}(3)}{n!} = -\frac{1}{45}\cdot\frac{1}{5^n} - \frac{(-1)^n}{36}\cdot\frac{1}{4^n}$$

and hence

$$f^{(n)}(3) = \left(-\frac{1}{9\cdot 5^{n+1}} - \frac{(-1)^n}{9\cdot 4^{n+1}}\right)n!$$

5. (7 pts each part, 14 pts total)

(a) Express the following function as a Maclaurin series

$$f(x) = \int_0^x \frac{1-\cos\sqrt{t}}{t}\,dt, \qquad x > 0.$$

$$1-\cos\sqrt{t} = 1 - \sum_0^\infty \frac{(-1)^n t^n}{(2n)!} = -\sum_{n=1}^\infty \frac{(-1)^n t^n}{(2n)!} = \sum_{n=1}^\infty \frac{(-1)^{n+1} t^n}{(2n)!}.$$

$$\frac{1-\cos\sqrt{t}}{t} = \sum_{n=1}^\infty \frac{(-1)^{n+1}\, t^{n-1}}{(2n)!}$$

$$f(x) = \int_0^x \frac{1-\cos\sqrt{t}}{t}\,dt = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{(2n)!}\cdot\frac{x^n}{n}.$$

(b) For $x = 0.1$, find a specific partial sum s_N of the series in part (a) for which $|f(0.1) - s_N| < 10^{-4}$.

$$f(0.1) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{(2n)!}\cdot\frac{(0.1)^n}{n}.$$

This is a strictly alternating series, with terms decreasing in absolute value and tending to zero as $n \to \infty$.

So if we use the first two terms of the series i.e.

$$S_N = \frac{1}{2}\cdot\frac{(0.1)^1}{1} - \frac{1}{4!}(0.1)^2$$

Then $|f(0.1) - S_N| < \frac{1}{6!}\frac{(0.1)^3}{3} = \frac{1}{2160}\left(\frac{1}{10}\right)^3 < \frac{1}{2000}\times\left(\frac{1}{10}\right)^3 = \frac{1}{2}10^{-6}.$

So the required specific partial sum is

$$S_N = \frac{1}{20} - \frac{1}{2400} = \frac{119}{2400}.$$

6. **(6 pts each part, total 12 pts)**

(a) Let $f(x) = \int_0^x \frac{1}{1+t^3}\,dt$. Find the Taylor series generated by f at $a = 0$.

If $0 \le t < 1$, then $0 \le t^3 < 1$ and

$$\frac{1}{1+t^3} = \sum_{n=0}^{\infty} (-1)^n (t^3)^n = \sum_{0}^{\infty} (-1)^n t^{3n}.$$

$$\therefore \quad f(x) = \int_0^x \frac{1}{1+t^3}\,dt = \sum_0^{\infty} \frac{(-1)^n \cdot x^{3n+1}}{3n+1}, \qquad 0 \le x < 1.$$

(b) Using the identity

$$\frac{1}{1+c} = 1 - c + c^2 - \cdots + (-1)^n c^n + \frac{(-1)^{n+1} c^{n+1}}{1+c},$$

prove that the series of part (a) converges to $f(x)$ for $0 \le x \le 1$.

Putting $c = t^3$ in the identity, we get

$$\frac{1}{1+t^3} = 1 - t^3 + t^6 - \cdots + (-1)^n t^{3n} + \frac{(-1)^{n+1} t^{3n+3}}{1+t^3}.$$

Integrating between 0 and x, $0 \le x \le 1$, get

$$f(x) = \int_0^x \frac{1}{1+t^3}\,dt = x - \frac{x^4}{4} + \frac{x^7}{7} - \cdots + \frac{(-1)^n x^{3n+1}}{3n+1} + \int_0^x \frac{(-1)^{n+1} t^{3n+3}}{1+t^3}\,dt$$

So the remainder is

$$R_n(x,0) = \int_0^x \frac{(-1)^{n+1} \cdot t^{3n+3}}{1+t^3}\,dt. \qquad 0 \le x \le 1.$$

$$\therefore \quad |R_n(x,0)| \le \int_0^x \frac{t^{3n+3}}{1+t^3}\,dt \le \int_0^x t^{3n+3}\,dt = \frac{x^{3n+4}}{3n+4} \le \frac{1}{3n+4}$$

So $\lim_{n\to\infty} R_n(x,0) = 0$ for each $0 \le x \le 1$.

Hence the series of part (a) converges to $f(x)$ for $0 \le x \le 1$

7. (5 pts each part, 10 pts total)

(a) If $0 < a_n < 1$, and $\sum_{n=1}^{\infty} a_n$ converges, find all values of p for which the series

$$\sum_{n=1}^{\infty}(a_n - \sin a_n)^p$$

converges.

Since $\sum_1 a_n$ converges, we have $\lim_{n\to\infty} a_n = 0$.

Applying L.C.T. with $\sum_1^{\infty}(a_n^3)^p$, we get

$$\lim_{n\to\infty} \frac{(a_n - \sin a_n)^p}{(a_n^3)^p} = \lim_{n\to\infty}\left(\frac{a_n - \sin a_n}{a_n^3}\right)^p = \lim_{x\to 0}\left(\frac{x-\sin x}{x^3}\right)^p$$

$$= \lim_{x\to 0}\left(\frac{x - (x - \frac{x^3}{3!} + \ldots)}{x^3}\right)^p = \left(\frac{1}{3!}\right)^p = \frac{1}{6^p}$$

$\therefore$ The given series converges for all values of p for which the series $\sum_1^{\infty} a_n^{3p}$ converges.

Since $0 < a_n < 1$, $a_n^{3p} < a_n$ if $p \geq \frac{1}{3}$ and $\sum_1^{\infty} a_n^{3p}$ converges.

So certainly the given series converges for all $p \geq \frac{1}{3}$.

But in addition, for a given series there ~~may~~ be an α, $0 \leq \alpha \leq 1$ such that $\sum a_n^{\beta}$ converges for $\beta > \alpha$ and diverges for $\beta < \alpha$. This α may be strictly less than 1. So our series converges for all $3p > \alpha$ or $p > \alpha/3$.

(b) If $a_n > 0$, $\sum_{n=1}^{\infty} a_n$ diverges, and $S_n = \sum_{k=1}^{n} a_k$, prove that the series $\sum_{n=2}^{\infty} \frac{a_n}{S_n S_{n-1}}$ converges.

Let $T_n = \sum_{k=2}^{n} \frac{a_k}{S_k S_{k-1}}$ be the n^{th} partial sum of required series.

Then

$$T_n = \sum_{k=2}^{n}\left(\frac{1}{S_{k-1}} - \frac{1}{S_k}\right) = \frac{1}{S_1} - \frac{1}{S_n}.$$

Since $\sum a_n$ diverges and $a_n > 0$, $\lim_{n\to\infty} S_n = \infty$, so

$$\lim_{n\to\infty} \frac{1}{S_n} = 0$$

$$\therefore \lim_{n\to\infty} T_n = \frac{1}{S_1} - 0 = \frac{1}{S_1}.$$

Hence the series $\sum_2^{\infty} \frac{a_n}{S_n S_{n-1}}$ converges and its sum is $\frac{1}{S_1}$.